Financial Modeling and Analysis
Course Code: MATH 242
Introduction
Module 2: Fixed Income Instruments

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Day 1: Introduction, Coupon Bonds

Outline

Interest rates

- Interest rate
  - A rate paid for borrowing money
  - Expressed as a percent of principal - amount of money borrowed - per unit of time, genrally per annum
- Time to maturity or term of the borrowing (specified contractually)
- Interest rate may be categorized by whether it varies over the term
  - **Fixed**: same value for all or part of the term
  - Floating/Variable: varies over the term by reference of an index

Fixed income instruments

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- Money Markets: short-term lending
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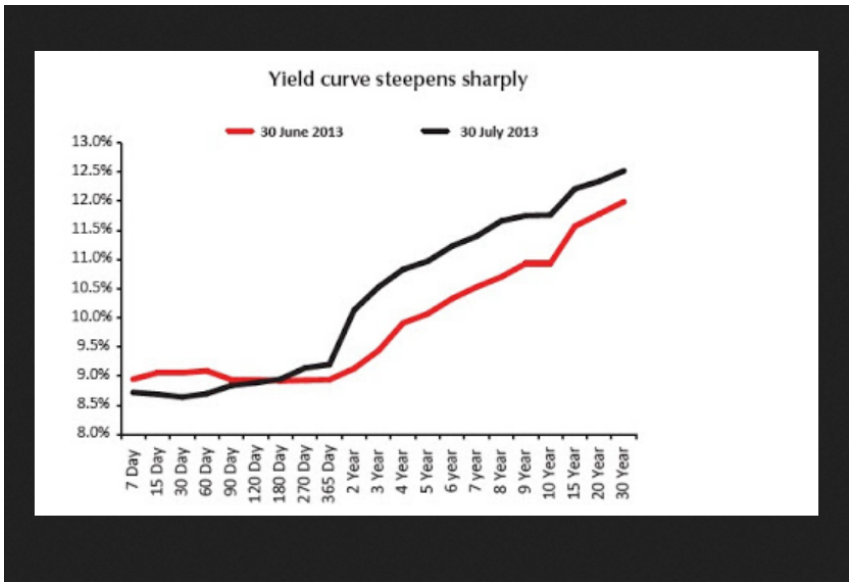
- Derivatives
  - Money market and bond futures
  - Interest rate swaps
  - Credit default swaps

Yield Curve

- Interest rates- risk-free or for a given obligor - are not a single risk factor
- Typically vary by the term of the loan
- Term structure of interest rates or yield curve: rates as function of maturity



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  - Particularly steep at very short end due to maturity
- But may be downward-sloping—yield curve inversion—overall or in some segments, e.g.
  - Short-term rates spike, not expected to persist, e.g. emerging-market rates under foreign exchange pressure
  - High demand for safe long-term bonds, e.g Pakistan Banao Certificates, National Savings Certificates, Khushali Savings Certificates

Equivalent ways of expressing the term structure

Term structure can be represented in different ways, each useful in certain contexts

- Zero-coupons and Coupons- curve and coupon rates (fixed, variable):
  - Money lent now and repaid at regular intervals or single specific time in the future
- Yield Curve: yields of bonds/interest as a function of term
- Forward curve and forward rates
  - Loans of a specified term to maturity commencing at different settlement date

Bonds

- A **bond** is a fixed-income security that pays the holder an amount, called the **face value** or **par value**, at a specified time, called the **maturity time**.
- If the face value is the only cash flow the bond generates, then the bond is called a **zero-coupon bond**
- A bond can generate cash during its life time. These are called **coupons** and can be thought of as interest rate payments. We will generally assume that each coupon payment is the same
- The interest rate used to discount the bond's payment is called the bond's **yield to maturity**, or **yield** for short

Bond ownership is like loaning money to a corporation
Are they risk free?

The general principle:

- Bond prices move in the opposite direction to interest rates
- Interest rate on the bond is fixed but the market interest rates fluctuate
- Market value of your bond fluctuates too

Zero Coupon Bonds

Pay no interest until maturity

- A 'zero' has a par/face value which is made to the bond holder at maturity
- It sells for less than the par value, therefore it is a 'discount' bond

General formula - Annual compounding

$$PRICE = PAR(1+r)^{-T}$$

General formula - Semi-Annual compounding

$$PRICE = PAR(1+\frac{r}{2})^{-2T}$$

Day 2-4: Coupon Bonds

Coupon Bonds

- make regular interest payments
- sell at par value when issued
- at maturity, one receives the principal and the final interest payment

General Formula for Semi-Annual Coupon Payment

$$
\begin{aligned}
PRICE &= \sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{PAR}{(1+r)^{2t}} \\
&= \frac{C}{r}\{1-(1+r)^{-2T}\} + \frac{PAR}{(1+r)^{2T}} \\
&= \frac{C}{r} - \{PAR - \frac{C}{r}\}(1+r)^{-2T}
\end{aligned}
$$

Bond Prices and Yield to Maturity

- Examples:
  - Short term issue: zero coupon maturing at 1 year
  - Coupon bonds maturing at 2, 3, 4 years
- Annualized interest rates, annual pay frequency and componding

term	coupon	price	yield
1	0.0	98.692	1.325
2	1.75	100.104	1.697
3	2.0	100.237	1.918
4	2.0	100.029	1.992

Bond price, yield and spot relationships

Data is generally given in the form of prices and/or yields

- Let $y_1, y_2, \cdots, y_T$ be the annual spot rates for coupon bonds of maturities $1, 2, \cdots, T$ years
- Price of a T-year bond with coupon payments C is related to its (generally observable) yield rate y

$$PRICE = \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \cdots + \frac{C}{(1+y)^T} + \frac{PAR}{(1+y)^T}$$

- Bond price is also related to its (unobservable) spot rates $y_1, y_2, \cdots y_T$:

$$PRICE = \frac{C}{(1+y_1)} + \frac{C}{(1+y_2)^2} + \cdots + \frac{C}{(1+y_T)^T} + \frac{PAR}{(1+y_T)^T}$$

Term Structure

The term structure of interest rates is a description of how, at a given time, yield to maturity depends on maturity. It can be described by any one of the following

- prices of zero-coupon bonds of maturities $1, 2, \cdots, n$ years, denoted by $P(1), P(2), \cdots, P(n)$
- spot rates (yields of maturity of zero-coupon bonds) of maturities $1, 2, \cdots, n$ years
- forward rates $r_1, \cdots, r_n$, where ri is the forward rate that can be locked in now for borrowing in the ith future year

Each of the above can be used to compute the other!

Term Structure

- Break down the time interval between present and maturity time into short time segments
- There is a constant interest rate within each time segment
- For example, a 2 year loan can be considered as two consecutive 1-year loans, or 4 half-year loans

Continuous Compounding

- Let's consider forward rates $r_1, r_2, \cdots, r_n$
- Simplifies the relationship between forward rates, yields to maturity and prices of zero coupon bonds.

$$P(n) = \frac{PAR}{e^{r_1+r_2+\cdots+r_n}}$$

- Also $log\{\frac{P(n-1)}{P(n)}\} = r_n$
- The yield to maturity of an n-year zero-coupon bond solves the equation

$$P(n) = \frac{PAR}{e^{ny_n}}$$

- It can be deduced

$$y_n = \frac{(r_1 + \cdots + r_n)}{n}.$$

Continuous forward rates

- Forward rates should be modeled as a function varying continuously in time, i.e. $r(t)$

$$PRICE = \frac{PAR}{exp\{\int_0^T r(t)dt\}} = \quad PAR\, exp\{-\int_0^T r(t)dt\}$$

$$=> \quad log(PRICE) = log(PAR) - \int_0^T r(t)dt$$

$$=> \quad -\frac{d}{dT} log\{PRICE\} = r(T) \forall T$$